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Constructing Linear Parameter Varying Models through Adaptation for the Control of a Class of Nonlinear Systems

Coşku Kasnakoğlu
Department of Electrical and Electronics Engineering
TOBB University of Economics and Technology
Söğütözü, Ankara 06560, Turkey
Email: kasnakoglu@etu.edu.tr

***Abstract*—In this paper a novel method is proposed for constructing linear parameter varying (LPV) system models through adaptation. For a class of nonlinear systems, an LPV model is built using its linear part, and its coefficients are considered as time-varying parameters. The variation in time is controlled by an adaptation scheme with the goal of keeping the trajectories of the LPV system close to those of the original nonlinear system. Using the LPV model as a surrogate, a dynamical controller is built by utilizing self-scheduling methods for LPV systems, and it is shown that this controller will indeed stabilize the original nonlinear system.**

I. INTRODUCTION

Linear parameter-varying (LPV) systems are an important class of dynamical systems, where the system model has a linear structure, but it is dependent on one or more parameters that are time-varying. An LPV system therefore represents a family of linear time varying (LTV) systems, one for each parameter trajectory [1]. LPV models can also be interpreted as a weighted combination of linear models, where the weights are the elements of the parameter vector. With this interpretation, one can also utilize LPV models to provide continuous local estimates of LTI models [2]. Since having one LPV model is much more compact than keeping an array of LTI models at different operating points, LPV models are of high interest for industrial applications where gain-scheduling approaches are common practice [3]. An additional benefit offered by LPV systems is the availability of systematic and robust controller design methods that can cope with arbitrary fast variations of the parameter vector [4]. Control designs based on LPV models have enjoyed a fair amount of success in the control of aircrafts [5], missiles [4], land vehicles [6], engines [7] power plant processes [8] and fluid flow problems [9].

Despite their usefulness, it is difficult to build LPV models in the first place, and methods to obtain such models is a field of active research. Typically one collects input, output and parameter trajectory data, and utilizes LPV system identification methods, among which one can list linear fractional transformations [10], subspace identification methods [11], least mean square and recursive least-squares algorithms [12], prediction error methods [5] and interval analysis techniques [13].

In this work we develop a novel and alternative method for building LPV system models, where the main idea is to start with a nonlinear dynamical model describing the system, and approximate this system with an LPV model whose parameter trajectories are generated by an adaptation scheme. For many real-life processes, considerable effort has already been devoted to their mathematical modeling, as a result of which there exist accurate nonlinear models readily available for these processes. While these models can describe the dynamics with considerable accuracy, their high complexity makes controller design extremely difficult. Obtaining an LPV model to approximate these models enables the use of the systematic controller design tools developed for LPV systems [4]. In this paper an LPV system is built from the linear portion of the original nonlinear system dynamics and its coefficients are regarded as parameters. An adaptation scheme is constructed to modify the parameters in time so that the response of the LPV system matches that of the original nonlinear system. A controller is designed using self-scheduled robust design approaches available for LPV systems and using results from adaptive and nonlinear control theories [14, 15], it is shown that this controller will indeed stabilize the original nonlinear dynamics.

II. PROBLEM DESCRIPTION

In this paper we consider nonlinear dynamical systems of the form

$$\dot{x} = Lx + L_{\mathrm{in}}u + \Phi_N(x,u) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $L \in \mathbb{R}^{n\times n}$, $L_{\mathrm{in}} \in \mathbb{R}^{n\times m}$, and $\Phi_N : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. The function Φ_N vanishes at $(x,u) = (0,0)$ together with its first derivatives, and satisfies a global Lipschitz condition of the form

$$\exists k_N \in \mathbb{R}_+ \text{ such that } \|\Phi_N(x,u)\| \le k_N\|\mathrm{col}(x,u)\| \;\forall x,u \tag{2}$$

where $\|\cdot\|$ is the Euclidian norm and col stands for column vector, i.e.

$$\mathrm{col}(v_1,v_2,\ldots,v_n) = [v_1^T \; v_2^T \; \ldots \; v_n^T]^T \;.$$

The first task is to design a linear parameter varying (LPV) system

$$\dot{\hat{x}}(t) = \hat{L}(t)\hat{x}(t) + \hat{L}_{\text{in}}(t)u(t) + \hat{L}_{\text{err}}(t)\left(\hat{x}(t) - x(t)\right) \quad (3)$$

which closely represents the system in (1); that is, if $e := \hat{x} - x$, then e should remain bounded and small as $t \to \infty$. The second task is to design a controller for this system that achieves the stabilization of the system, as well as keeping the effect of the disturbance caused by the error e within reasonable limits. In addition, it is necessary to show that this controller will succeed in stabilizing the original system (1). The next section (Section III) will be concerned with these two tasks.

III. LPV MODELLING AND CONTROL DESIGN

In this section an LPV system of the form (3) is built to approximate the nonlinear system (1). Let us first build the parameter vector $\theta_L \in R^p$ as

$$\theta_L := \text{col}\left(L(:), L_{\text{in}}(:)\right) \quad (4)$$

where $L(:)$ denotes the column vector formed by stacking all elements of L on top of each other, i.e.

$$L(:) := \text{col}\left(L_{11}, L_{21}, \ldots, L_{n1}, \ldots, L_{n1}, L_{n2}, \ldots L_{nn}\right) \ . \quad (5)$$

The column vector $L_{\text{in}}(:)$ is defined similarly, and p is the total number of coefficients in L and L_{in}. Let us also define $\Phi_L \in \mathbb{R}^n \times \mathbb{R}^m$ to satisfy the expression

$$\Phi_L(x,u)\theta_L = Lx + L_{\text{in}}u \ . \quad (6)$$

In other words, $\Phi(x,u)$ is a $n \times p$ dimensional matrix with elements $\{\Phi_L(x,u)_{ij} \mid i = 1, \ldots, n,\ j = 1, \ldots, p\}$ where $\Phi_L(x,\gamma)_{ij}$ denotes the element at row i and column j. For instance, it is clear from (4) that the second parameter of θ_L is the second parameter of L, which is seen to be L_{21} from (5). Also, the second element of the state vector $x = \text{col}(x_1, x_2, x_3, \ldots, x_n)$ is x_2. Then from (6), it is clear that $\Phi_L(x,u)_{22} = x_1$. Other elements of $\Phi_L(x,u)$ can be constructed similarly. For future reference we also note that

$$\|\Phi_L(x,u)\theta_L\| \leq \|L\|\|x\| + \|L_{\text{in}}\|\|u\| \leq k_L\|\text{col}(x,u)\| \quad (7)$$

where $k_L := 2\max\{\|L\|, \|L_{\text{in}}\|\}$. With the definition of $\Phi_L(x,u)$ as above, the LPV model sought can be written as

$$\dot{\hat{x}} = \Phi_L(x,u)\hat{\theta}_L - ke = \hat{L}x + \hat{L}_{\text{in}}u - ke \quad (8)$$

whose parameter vector $\hat{\theta}_L$ will be modified by an adaptation mechanism so that the state trajectory of (8) matches that of the nonlinear system (1). It should be emphasized that it is not the goal to achieve $\hat{\theta}_L \to \theta_L$; this is in fact undesirable, since it would imply that (8) approximates the behavior of (1) around only the origin $x = 0$. We would instead like θ_L to be modified to force the state trajectory of (8) to that of (1). In other words, the goal is to minimize the error $e = \hat{x} - x$, which is governed by the following dynamics [1]

$$\dot{e} = \dot{\hat{x}} - \dot{x} = \Phi_L(x,u)\hat{\theta}_L - ke - \Phi_L(x,u)\theta_L - \Phi_N(x,u) \ . \quad (9)$$

The adaptation mechanism considered for this purpose is of the following form

$$\dot{\hat{\theta}}_L = -k_t\Phi_L^T(x,u)e - k_t\Psi(e,\tilde{\theta}_L,x,u) \quad (10)$$

where $\tilde{\theta}_L := \hat{\theta}_L - \theta_c$, the function Ψ is defined as

$$\Psi(e,\tilde{\theta}_L,x,u) := \begin{cases} 0, & \|\text{col}(e,\tilde{\theta}_L)\| < k_x\|\text{col}(x,u)\|; \\ k_d\tilde{\theta}_L, & \|\text{col}(e,\tilde{\theta}_L)\| \geq k_x\|\text{col}(x,u)\| \end{cases} \quad (11)$$

and $k, k_t, k_d \in \mathbb{R}_+$, $\theta_c \in \mathbb{R}^p$ are constants to be selected as part of the design process. We also note the following for future reference: If L_c and $L_{\text{c,in}}$ are the matrices whose coefficients form θ_c, i.e. $\theta_c = \text{col}\left(L_c(:), L_{\text{c,in}}(:)\right)$, then it holds that

$$\|\Phi_L(x,u)\theta_c\| \leq \|L_c\|\|x\| + \|L_{\text{c,in}}\|\|u\| \leq k_c\|\text{col}(x,u)\| \quad (12)$$

where $k_c := 2\max\{\|L_c\|, \|L_{\text{c,in}}\|\}$. It can be shown that, by using the adaptation mechanism (10) with properly selected values for its constants, the error $e = \hat{x} - x$ can be made to remain bounded and small. This means that the state trajectories of the system (8) will approach those of the nonlinear system (1). The proof of this statement is postponed until Theorem 2; however we note that if this is the case, then the following interpretation can be made: Let us rearrange (8) as

$$\begin{aligned} \dot{\hat{x}} &= \hat{L}x + \hat{L}_{\text{in}}u - ke \\ &= \hat{L}(\hat{x} - e) + \hat{L}_{\text{in}}u - ke \\ &= \hat{L}\hat{x} + \hat{L}_{\text{in}}u + \hat{L}_{\text{err}}e \ . \end{aligned} \quad (13)$$

where $\hat{L}_{\text{err}} = -(\hat{L} + kI)$. One can then observe that (13) is of the same form as (3). Thus, if the signal e is bounded and small, one can regard system (8) as a linear parameter-varying system that approximates the original system, with the signal e entering as an external disturbance. The next task is to design a controller for this model to stabilize the system and also limit the effect of the error term on the dynamics. We will design a controller that can achieve these goals based on a robust automatic scheduling method [4], a brief summary of which is provided below.

Consider the following affine linear parameter dependent plant

$$\dot{x} = A(\theta)x + B_1(\theta)w + B_2u \quad (14)$$

$$z = C_1(\theta)x + D_{11}(\theta)w + D_{12}u \quad (15)$$

$$y = C_2x + D_{21}w \quad (16)$$

[1]The system (8) can also be though of an adaptive psuedo-observer; the prefix *psuedo* is due to the fact that that a real observer reconstructs the states from outputs, which is not the case here.

where x is the state, u is the control input, w is the disturbance input, y is the signal available for control, and z is the output to be controlled. The parameter vector θ is available in in real-time and varies in a *polytope* Θ of vertices $\theta_1, \dots, \theta_p$; i.e. $\theta \in \Theta$ where $\Theta := \mathrm{Co}\{\theta_1, \dots, \theta_p\} := \{\sum_{i=1}^{p} \alpha_i \theta_i : \alpha_i \geq 0, \sum_{i=1}^{p} \alpha_i = 1\}$ and Co stands for convex hull. We assume that $(A(\theta), B_2)$ is quadratically stabilizable over θ and $(A(\theta), C_2)$ is quadratically detectable over θ. The goal is to design a dynamic controller whose input is y and generates u which stabilizes the system (14)-(16) while minimizing the gain from w to z. For this purpose a linear parameter dependent controller having the following structure is considered

$$\dot{\zeta} = A_K(\theta)\zeta + B_K(\theta)y \tag{17}$$

$$u = C_K(\theta)\zeta + D_K(\theta)y \ . \tag{18}$$

To design this controller we utilize the results summarized in the theorem below.

Theorem 1. *Consider the LPV system in (14)-(16) and the dynamic controller structure given in (17)-(18). Let the controller matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ be chosen in the following way:*

1) *Find a matrix $X_{cl} = X_{cl}^T > 0$, and controller matrices A_{Ki}, B_{Ki}, C_{Ki}, D_{Ki} such that the matrices*

$$\begin{bmatrix} A_{cl}(\theta_i)^T X_{cl} + X_{cl} A_{cl}(\theta_i) & X_{cl} B_{cl}(\theta_i) & C_{cl}^T(\theta_i) \\ B_{cl}^T(\theta_i) X_{cl} & -\gamma I & D_{cl}^T(\theta_i) \\ C_{cl}(\theta_i) & D_{cl}(\theta_i) & -\gamma I \end{bmatrix} \tag{19}$$

are negative-definite for $i = 1 \dots p$.

2) *For a given value of θ, compute the matrices $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ defining the LPV controller as*

$$\begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix} = \sum_{i=1}^{p} \alpha_i(\theta) \begin{bmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{bmatrix}$$

where A_{cl}, B_{cl}, C_{cl}, D_{cl} are the closed loop system matrices, and $\alpha = (\alpha_1, \dots, \alpha_p)$ is a convex decomposition of θ such that $\theta = \sum_{i=1}^{p} \alpha_i \theta_i$ and $\sum_{i=1}^{p} \alpha_i = 1$.

Then, the closed loop system in is stable, with the stability established by the function $V_a(x_a) = x_a^T X_{cl} x_a$, where x_a is the system state. Moreover, it holds that $\|z\|_2 < \gamma \|w\|_2$.

Proof: See Apkarian et al. [4]. ■

For the problem at hand, the system to be controlled is given in (13), where the input to the controller is taken to be $y = x = \hat{x} - e$, and the system output is taken to be the $z = \mathrm{col}(x, u) = \mathrm{col}(\hat{x} - e, u)$. The system as such is rewritten below for later reference

$$\dot{\hat{x}} = \hat{L}(\hat{\theta}_L)\hat{x} - \left(\hat{L}(\hat{\theta}_L) + kI\right)e + \hat{L}_{\mathrm{in}}(\hat{\theta}_L)u \tag{20}$$

$$z = \begin{bmatrix} I \\ 0 \end{bmatrix}\hat{x} - \begin{bmatrix} I \\ 0 \end{bmatrix}e + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \tag{21}$$

$$y = \hat{x} - e \tag{22}$$

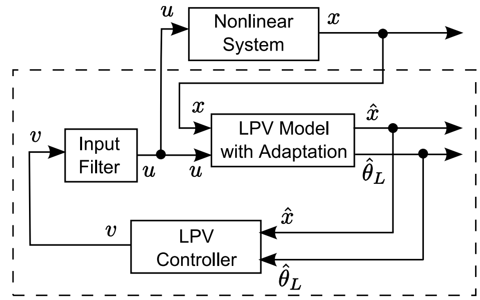


Fig. 1. Block diagram of the entire system.

where the dependence of $\hat{L}$ and $\hat{L}_{\mathrm{in}}$ on the parameters $\hat{\theta}_L$ have been shown explicitly. The goal is to design a controller of the form (17)-(18), that will stabilize the system and also minimize the effect of the error to the system, for all permissable parameter trajectories $\hat{\theta}_L(t)$. Since the parameters $\hat{\theta}_L$ come from the adaptation mechanism (10), they are available in real-time and therefore can be used for the automatic scheduling of the controller. The control design however, is not a straightforward application of Theorem 1 above, since in (20), the input vector u enters the dynamics through a coefficient $\hat{L}_{\mathrm{in}}(\hat{\theta}_L)$, which is dependent on the parameter vector $\hat{\theta}_L$. This prevents one from using Theorem 1 directly since the coefficient of the input, indicated as B_2 in (14), is assumed to be constant. The issue can be resolved by adding a known input filter to the system as follows

$$\dot{\xi} = A_u\xi + B_u v \tag{23}$$

$$u = C_u\xi \tag{24}$$

where ξ is the filter state and v is the filter input, which will be considered as the control input from this point on. Hence, if we augment the system (20)-(22) with this filter we get

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \hat{L}(\hat{\theta}_L) & \hat{L}_{\mathrm{in}}(\hat{\theta}_L)C_u \\ 0 & A_u \end{bmatrix}\begin{bmatrix} \hat{x} \\ \xi \end{bmatrix} - \begin{bmatrix} \hat{L}(\hat{\theta}_L) + kI \\ 0 \end{bmatrix}e + \begin{bmatrix} 0 \\ B_u \end{bmatrix}v \tag{25}$$

$$z = \begin{bmatrix} I & 0 \\ 0 & C_u \end{bmatrix}\begin{bmatrix} \hat{x} \\ \xi \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix}e \tag{26}$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \xi \end{bmatrix} - e \ . \tag{27}$$

In the augmented system (25)-(27), the coefficient of the input v is not dependent on the parameter vector $\hat{\theta}_L$. Thus, it is now possible to design an LPV controller that is automatically scheduled based on the parameter vector $\hat{\theta}_L$, through the procedure given in Theorem 1. Figure 1 shows a block diagram of the entire system including the nonlinear system, the input filter, the adaptation mechanism, and the controller.

IV. CONVERGENCE AND STABILITY ANALYSIS

The following theorem justifies the validity of the approach considered, by proving that 1) the adaptation scheme will force the LPV system state trajectories to converge to the state trajectories of the original nonlinear system, and 2) the LPV controller designed for this LPV system will also stabilize the original nonlinear system.

Theorem 2. *Consider the LPV plant augmented with an input filter, as given in (25)-(27). For this system, let an automatically scheduled LPV controller of the form*

$$\dot{\zeta} = A_K(\hat{\theta}_L)\zeta + B_K(\hat{\theta}_L)y \tag{28}$$

$$u = C_K(\hat{\theta}_L)\zeta + D_K(\hat{\theta}_L)y \tag{29}$$

be designed through the procedure given in Theorem 1 where the parameter vector $\hat{\theta}_L$ is determined by the dynamics given in (10), and k is chosen such that

$$k > \frac{1}{2}\left(k_L + k_N + k_c\right)^2 + \gamma^2 \ . \tag{30}$$

where k_N, k_L and k_c are as given in (2), (7) and (12). Then:

1) *The trajectories of the LPV system, with the parameter vector modified through the adaptation mechanism (10), will converge to the trajectories of the original nonlinear system (1).*
2) *The control signal u produced by the controller (28)-(29) will asymptotically stabilize the original nonlinear system (1).*

Proof: See Appendix A. ■

V. EXAMPLE

As an example we consider the following system

$$\dot{x}_1 = x_1 - 2x_2 + u + 3\tanh(x_2) + \sin(x_1 u) + \sqrt{|x_1 x_2| + 1} - 1 \tag{31}$$

$$\dot{x}_2 = 3x_1 + 2x_2 + 4u + \sin(x_1 x_2) + ue^{-u^2} \tag{32}$$

which is of the form given in (1). It can be shown that (2) is satisfied with Lipschitz constant $k_N = 9.1344$, the origin $(x_1, x_2) = (0, 0)$ is an unstable equilibrium under no forcing, and from non-zero initial conditions the trajectories diverge. The goal is to find u so as to stabilize the system and drive the state $x := \mathrm{col}(x_1, x_2)$ to zero. For this purpose, first an LPV system of the form (8) was obtained where

$$\hat{\theta}_L = \mathrm{col}(\hat{a}_{11}, \hat{a}_{12}, \hat{a}_{21}, \hat{a}_{22}, \hat{b}_1, \hat{b}_2) \tag{33}$$

$$\Phi_L(x, u) = \begin{bmatrix} x_1 & x_2 & 0 & 0 & u & 0 \\ 0 & 0 & x_1 & x_2 & 0 & u \end{bmatrix} \tag{34}$$

and $\hat{\theta}_L$ is to be dictated by an adaptation scheme of the form (10). To determine an estimate for the range of values in which the parameter vector $\hat{\theta}_L$ will vary, a high number input signals of various types including ramp functions, sine functions, chirp functions, square waves and white noise were applied to the system. Observing the values assumed by the parameters under these excitation signals, the polytope Θ such that $\hat{\theta}_L \in \Theta$ was chosen to be the 6-dimensional box

$$\Theta = \{0.32 < \hat{a}_{11} < 1.39,\ -2.51 < \hat{a}_{12} < -0.13,\ 1.88 < \hat{a}_{21} < 4.13, -2.21 < \hat{a}_{22} < 3.54,\ 0.49 < \hat{b}_1 < 1.74,\ 2.57 < \hat{b}_2 < 5.04\} \ . \tag{35}$$

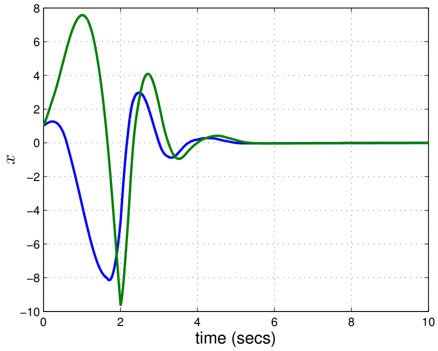
The constants of the adaptation mechanism (10) were selected as $k = 900$, $k_t = 100$, $k_d = 1$, $k_x = 20$ and $\theta_c = \mathrm{col}(0.855,\ -1.320,\ 3.005,\ 0.665,\ 1.115,\ 3.805)$ based on the discussions in Section IV. The controller was designed as described in Section III using the functions of the Robust Control Toolbox in MATLAB, and the quadratic $\mathcal{H}_\infty$ performance for the closed loop system from e to y was found to be $\gamma = 0.0311$. This implies that the error cannot energize, i.e. disturb the output more than a limited amount. The next step is to build, implement and test the full system shown in Figure 1. Figures 2-3 show the numerical simulation results for this configuration. For test purposes, the control action is set to be zero until $t = 2$ seconds, so that the system runs in open loop for this period. The controller is incorporated into the system by closing the loop at $t = 2$ seconds. It can be seen from Figure 2(b) that the controller achieves the desired stabilization of the LPV system and drives $\hat{x} \to 0$. Figure 2(c) shows the adaptation error $e = \hat{x} - x$, which seems to remain of the order 10^{-3}. The fact that $\hat{x} \to 0$ also implies that $x \to e$, and since error is very small, this practically means that the $x \to 0$ as well, as confirmed by Figure 2(a). Figure 3 shows the parameter vector $\hat{\theta}_L$ generated by the adaptation mechanism. It can be seen that there are considerable variations in the parameter trajectories throughout the process. Nevertheless, since these parameters are estimated internally by the adaptation mechanism and are available to the controller in real-time, the controller can utilize the current value of the parameter vector $\hat{\theta}_L(t)$ to automatically schedule its matrices and hence succeeds in the desired stabilization.

VI. CONCLUSIONS AND FUTURE WORKS

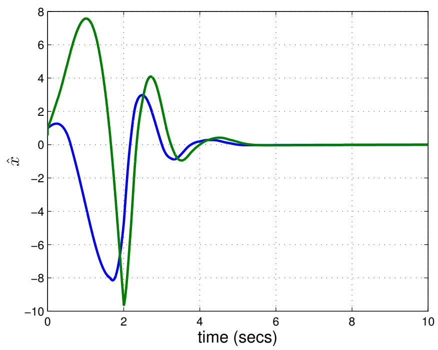
In this work a new method is proposed for building LPV system models through adaptation, for a class of nonlinear systems. Starting from the nonlinear system dynamics, an LPV model was built using the linear part, and its coefficients were considered as time-varying parameters. An adaptation scheme was constructed to control the variation of the parameters in time, with the goal of keeping the trajectories of the LPV system close to those of the original nonlinear system. Using the LPV model as a surrogate, a dynamical controller was built using robust self-scheduling methods for LPV systems, and it is shown that this controller would indeed stabilize the original nonlinear system. The results were illustrated on an example system where it was seen that the controller design based on the LPV system is successful in achieving the desired stabilization.

The main contribution of the paper is to illustrate a novel approach for obtaining LPV models from a class of nonlinear systems through adaptation techniques. It is also shown how this model can be used to design an automatically-scheduled robust controller by utilizing results for LPV systems, and it is proved that this controller will indeed stabilize the original nonlinear system.

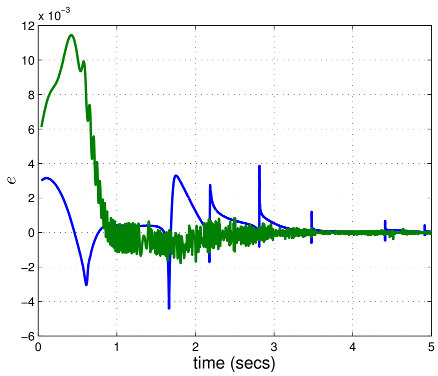
Future research directions include the expansion of the results to other classes of nonlinear systems, estimating the parameter polytope in real-time, using different adaptation



(a) System state x



(b) Adapted state $\hat{x}$



(c) Error between the adapted state $\hat{x}$ and the system state x

Fig. 2. States of the flow system and adapted states with the controller turned on at $t = 2$ seconds.

laws and control techniques, and application of the approach developed to physical problems experiments.

Acknowledgements

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Appendix A
Proof of Theorem 2

Since the controller is computed for system (25)-(27) using the approach outlined in Theorem 1, it holds that

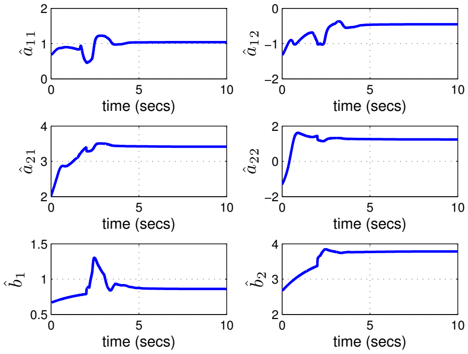

Fig. 3. Parameter vector $\hat{\theta}_L$ with the controller turned on at $t = 2$ seconds.

$\|\mathrm{col}(x,u)\|_2 < \gamma\|e\|_2$, so the closed loop system is strictly dissipative with a supply rate

$$q(e,x,u) = \gamma^2\|e\|^2 - \|\mathrm{col}(x,u)\|^2 \tag{36}$$

and storage function

$$V_a(x_a) = x_a^T X_{cl} x_a \quad . \tag{37}$$

That is,

$$\dot{V}_a \leq -\mu_a\|x_a\|^2 + q(e,a,u) \tag{38}$$

where $\mu_a \in \mathbb{R}_+$, X_{cl} is as given in the statement of Theorem 1, and $x_a := \mathrm{col}(\hat{x},\xi,\zeta)$ is the augmented state vector containing the states of the LPV system, the input filter and the controller. Note that

$$\underline{\alpha}_a(\|x_a\|) \leq V_a(x_a) \leq \overline{\alpha}_a(\|x_a\|) \tag{39}$$

where $\underline{\alpha}_a(r) := \lambda_{\min} r^2$, $\overline{\alpha}_a(r) := \lambda_{\max} r^2$ and $\lambda_{\min}$, $\lambda_{\max}$ are the smallest and largest eigenvalues of X_{cl}. Define

$$V_t(e,\hat{\theta}_L) := \frac{1}{2}e^T e + \frac{1}{2k_t}\tilde{\theta}_L^T\tilde{\theta}_L \tag{40}$$

and note that

$$\underline{\alpha}_t(\|\mathrm{col}(e,\tilde{\theta}_L)\|) \leq V_t(e,\tilde{\theta}_L) \leq \overline{\alpha}_t(\|\mathrm{col}(e,\tilde{\theta}_L)\|) \tag{41}$$

where $\underline{\alpha}_t(r) := k_2 r^2$, $\overline{\alpha}_t(r) := k_3 r^2$ and

$$k_2 := \min\left\{\frac{1}{2}, \frac{1}{2k_t}\right\} \tag{42}$$

$$k_3 := \max\left\{\frac{1}{2}, \frac{1}{2k_t}\right\}. \tag{43}$$

Consider now the entire system including the LPV plant, input filter, controller, adaptation law and the original nonlinear model, which is an autonomous system (Figure 1). Consider the state vector $x_e := \mathrm{col}(\hat{x},\xi,\zeta,e,\tilde{\theta}_L)$ for the entire system. Note that the state of the original nonlinear system is included implicity since $x = \hat{x} - e$. Consider a candidate Lyapunov function

$$V(x_e) := V_a(\hat{x},\xi,\zeta) + V_t(e,\tilde{\theta}_L) \tag{44}$$

where V_a is as defined in (37) and V_t is as in (40). Note that

$$\underline{\alpha}_e(\|x_e\|) \leq V(x_e) \leq \overline{\alpha}_e(\|x_e\|) \tag{45}$$

where $\underline{\alpha}_e(r) = k_4 r^2$, $\overline{\alpha}_e(r) = k_5 r^2$ and

$$k_4 := \min\{\lambda_{\min}, k_2\} \tag{46}$$

$$k_5 := \max\{\lambda_{\max}, k_3\} \ . \tag{47}$$

Differentiating (44) along trajectories yields

$$\dot{V}(x_e) = \dot{V}_a(\hat{x}, \xi, \zeta) + \dot{V}_t(e, \tilde{\theta}_L) \tag{48}$$

where we know that $\dot{V}_a$ satisfies (38). To obtain a bound for $\dot{V}_t$, note from (40) that

$$\begin{aligned}
\dot{V}_t &= e^T \dot{e} + \frac{1}{k_t}\tilde{\theta}_L^T \dot{\tilde{\theta}}_L \\
&= e^T \left(\Phi_L(x,u)\hat{\theta}_L - ke - \Phi_L(x,u)\theta_L - \Phi_N(x,u)\right) \\
&+ \frac{1}{k_t}\tilde{\theta}_L^T \left(-k_t \Phi_L^T(x,u)e - k_t k_d \tilde{\theta}_L\right) \\
&= e^T \Phi_L(x,u)\hat{\theta}_L - k\|e\|^2 - e^T \Phi_L(x,u)\theta_L - e^T \Phi_N(x,u) \\
&- \tilde{\theta}_L^T \Phi_L^T(x,u)e - \tilde{\theta}_L^T \Psi(e, \tilde{\theta}_L, x, u) \\
&\leq -k\|e\|^2 + \|e\|\|\Phi_L(x,u)\theta_L\| + \|e\|\|\Phi_N(x,u)\| \\
&+ \|e\|\|\Phi_L(x,u)\theta_c\| - \tilde{\theta}_L^T \Psi(e, \tilde{\theta}_L, x, u) \\
&\leq -k\|e\|^2 + \|e\| \|\mathrm{col}(x,u)\| (k_L + k_N + k_c) \\
&- \tilde{\theta}_L^T \Psi(e, \tilde{\theta}_L, x, u) \\
&\leq -k\|e\|^2 + \frac{1}{2}\|e\|^2 (k_L + k_N + k_c)^2 + \frac{1}{2}\|\mathrm{col}(x,u)\|^2 \\
&- \tilde{\theta}_L^T \Psi(e, \tilde{\theta}_L, x, u)
\end{aligned}$$

where we have used (2), (7), (12) and Young's inequality[2] as needed. Collecting similar terms and using the fact that $\tilde{\theta}_L^T \Psi(e, \tilde{\theta}_L, x, u) \geq 0$ yields

$$\begin{aligned}
\dot{V}_t \leq &-\left(k - \frac{1}{2}(k_L + k_N + k_c)^2\right)\|e\|^2 + \frac{1}{2}\|\mathrm{col}(x,u)\|^2 \\
&- \tilde{\theta}_L^T \Psi(e, \tilde{\theta}_L, x, u) \qquad (49) \\
\leq &-(k - k_6)\|e\|^2 + \frac{1}{2}\|\mathrm{col}(x,u)\|^2 \qquad (50)
\end{aligned}$$

where

$$k_6 := \frac{1}{2}(k_L + k_N + k_c)^2 \tag{51}$$

Substituting (38) and (50) into (48) yields

$$\begin{aligned}
\dot{V}(x_e) &= \dot{V}_a(\hat{a}, \xi, \zeta) + \dot{V}_t(e, \tilde{\theta}_L) \\
&\leq -\mu\|x_a\|^2 + \gamma^2\|e\|^2 - \|\mathrm{col}(x,u)\|^2 \\
&- (k - k_6)\|e\|^2 + \frac{1}{2}\|\mathrm{col}(x,u)\|^2 \\
&\leq -\mu\|x_a\|^2 + \left(k - k_6 - \gamma^2\right)\|e\|^2 \qquad (52)
\end{aligned}$$

which is negative if (30) holds. Hence, all trajectories of the system are bounded, $x_a = \mathrm{col}(\hat{x}, \xi, \zeta) \to 0$ and $e \to 0$. The fact that $e = \hat{x} - x \to 0$ implies $\hat{x} \to x$, which states that the trajectories of the LPV system, whose parameter variations are controlled by the designed adaptation mechanism, will converge to those of the original nonlinear system. The fact that $x_a \to 0$ implies $\hat{x} \to 0$, and since $\hat{x} \to x$, we have $x \to 0$, which states that the LPV control design based on the LPV plant is indeed successful in asymptotically stabilizing the origin of the nonlinear model. □

[2]Let $x, y, \varepsilon \in \mathbb{R}_+$, then $xy \leq \frac{x^2}{2\varepsilon} + \frac{\varepsilon y^2}{2}$.

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